Preservation of Total Nonnegativity under the Hadamard Product and Related Topics

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Abstract. We present various results related to our recent result that the Hadamard (i.e. coefficient–wise) product of two Hurwitz stable polynomials is again Hurwitz stable, resp. that the set of the nonsingular totally nonnegative Hurwitz matrices is closed under the Hadamard product.

§1. Introduction

The *Hadamard product* of two polynomials

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{1}$$
$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \tag{2}$$

in $\mathbb{R}[x]$ is defined to be

$$(p * q)(x) = a_k b_k x^k + a_{k-1} b_{k-1} x^{k-1} + \cdots + a_1 b_1 x + a_0 b_0$$

where $k = \min(n, m)$.

A polynomial $p \in \mathbb{R}[x]$ is termed *Hurwitz* (or *asymptotically*) *stable* if every zero of p is in the open left half of the complex plane, it is *Schur stable* if every zero of p is in the open unit disc centered around 0.

In a recent paper [7] we have shown that the Hadamard product of two Hurwitz stable polynomials is again Hurwitz stable, whereas the Hadamard product of two Schur stable polynomials need not be Schur stable. Related results on weak Hurwitz stability can also be found in [7].

Analogous statements for the coefficient–wise sum or quotient are not true: the coefficient–wise sum of two Hurwitz (resp. Schur) stable polynomials need not be stable, cf. [3] for the Hurwitz and e.g. [1] for the Schur case, and the coefficient–wise quotient of two Hurwitz (resp. Schur) stable polynomials need not be stable. (This is simply seen by taking a stable polynomial

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of n-th degree with only nonzero coefficients; then dividing this polynomial coefficient–wise by itself results in the polynomial

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$$

which has all its zeros on the unit circle.)

The purpose of this paper is to show that the preservation of Hurwitz stability under the Hadamard product is restricted to polynomials with real coefficients and to present related results. These include

in the polynomial case negative results on the related Schur product;

in the matrix case

- a list of all subclasses of totally nonnegative matrices known to us to be closed under the Hadamard (i.e. coefficient–wise) product,
- a negative result on the preservation of total nonnegativity under the Hadamard product for the set of the nonsingular triangular finite Toeplitz matrices (in contrast to the infinite case),
- results on the Hadamard product of two stable matrices.

§2. Polynomial Results

First, we show that the set of the complex Hurwitz stable polynomials is not closed under the Hadamard product. This applies if the multiplication of two (complex) coefficients is performed conventionally as shown in the following example.

Example 1. Let

$$p(x) = (x+1)(x+1-2i) = x^2 + 2(1-i)x + 1 - 2i.$$

Then the Hadamard product of p with itself is

$$x^2 - 8ix - 3 - 4i$$

which is not Hurwitz stable.

Also, it applies if the multiplication of two coefficients is performed in the following way ($\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$):

$$(\alpha_1 + \beta_1 i)(\alpha_2 + \beta_2 i) = \alpha_1\alpha_2 + \beta_1\beta_2 i.$$

This is shown in the following example.

Example 2. Let

$$p(x) = x^2 + x + 1$$
$$q(x) = (x + 1/2 + i/2)^2 = x^2 + (1+i)x + i/2$$

which are Hurwitz stable. Then the Hadamard product of p with q is $x^2 + x$ which is not Hurwitz stable.

A product related to the Hadamard product is the *Schur product* of two polynomials p, q given by (1), (2)

$$(p \odot q)(x) = k!\, a_k b_k x^k + (k-1)!\, a_{k-1} b_{k-1} x^{k-1} + \cdots + 1!\, a_1 b_1 x + 0!\, a_0 b_0.$$

This product was considered e.g. in [4,18]. The following examples show that the set of the Hurwitz (resp. Schur) stable polynomials is not closed under the Schur product.

Example 3. Let

$$p(x) = 3x^3 + 2x^2 + 2x + 1$$

which is Hurwitz stable. But

$$(p \odot p)(x) = 54x^3 + 8x^2 + 4x + 1$$

is not stable.

Example 4. Let

$$p(x) = x^2 - 1.7x + 0.72 = (x - 0.9)(x - 0.8)$$

which is Schur stable. But

$$(p \odot p)(x) = 2x^2 + 2.89x + 0.5184$$

has a zero at $-1.23\ldots$.

§3. Matrix Results

We now turn to some matrix results related to stability and Hadamard products of polynomials. With a real polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ we associate the n–by–n *Hurwitz matrix* $H(p) = (h_{ij}(p))$, defined by $h_{ij}(p) = a_{2j-i}$ for each $1 \le i,\ j \le n$, where by convention $a_k = 0$ if $k < 0$ or $k > n$.

A real matrix M is *totally nonnegative* if every minor of M is nonnegative [6,14]. It was shown in [2,15] that the Hurwitz matrix $H(p)$ is nonsingular and totally nonnegative if and only if p is Hurwitz stable and $a_n > 0$.

Given two n–by–n matrices $A = (a_{ij})$ and $B = (b_{ij})$ the *Hadamard product* of A and B is the n–by–n matrix $A * B$ defined by $A * B = (a_{ij} b_{ij})$. Comprehensive surveys of the Hadamard matrix product are found in [11,12]. As shown in [13,16], the Hadamard product of two totally nonnegative matrices need not be totally nonnegative. However, some subclasses of totally nonnegative matrices are known to be closed under Hadamard multiplication. These include:

(i) generalized Vandermonde matrices $(x_i^{\alpha_j})$ with $1 \le i,\, j \le n$, where either the bases $0 < x_1 < x_2 < \cdots < x_n$ or the exponents $\alpha_1 < \cdots < \alpha_n$ are fixed (see p. 99 of [5]);

(ii) tridiagonal totally nonnegative matrices [16];

(iii) triangular totally nonnegative infinite Toeplitz matrices such that the value on the k-th diagonal is a polynomial function of k [17];

(iv) totally nonnegative Green's matrices (g_{ij}) with $g_{ij} = a_{\min(i,j)} b_{\max(i,j)}$, where $a_1, b_1, \ldots, a_n, b_n$ are positive real numbers (this fact follows from p. 91 of [6] and p. 111 of [14]);

(v) finite moment matrices [8,9] of probability measures which are either symmetric around 0 or possess nonnegative support [10];

(vi) nonsingular totally nonnegative Hurwitz matrices [7].

Remark. In contrast to the infinite case, the set of the nonsingular totally nonnegative upper (resp. lower) triangular (finite) Toeplitz matrices is not closed under the Hadamard product:

Example 5. Let

$$A = \begin{pmatrix} 0.5 & & & & \\ 1 & 0.5 & & \mathbf{0} & \\ 1 & 1 & 0.5 & & \\ 1 & 1 & 1 & 0.5 & \\ 1 & 1 & 1 & 1 & 0.5 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & & & & \\ 4 & 1 & & \mathbf{0} & \\ 9 & 4 & 1 & & \\ 16 & 9 & 4 & 1 & \\ 25 & 16 & 9 & 4 & 1 \end{pmatrix}.$$

Both matrices are nonsingular and totally nonnegative. But

$$A * B = \begin{pmatrix} 0.5 & & & & \\ 4 & 0.5 & & \mathbf{0} & \\ 9 & 4 & 0.5 & & \\ 16 & 9 & 4 & 0.5 & \\ 25 & 16 & 9 & 4 & 0.5 \end{pmatrix}$$

has a 3–by–3 submatrix with negative determinant in the lower left corner.

We now turn to the question of preservation of matrix stability under the Hadamard product. A matrix is termed *Hurwitz* (resp. *Schur*) *stable* if its characteristic polynomial is Hurwitz (resp. Schur) stable. The Hadamard product of two Hurwitz stable matrices need not be Hurwitz stable already in the trivial case $n = 1$. Also, the set of the Schur stable matrices is not closed under the Hadamard product:

Example 6. Let A be the companion matrix of the polynomial p of Ex. 4, i.e.

$$A = \begin{pmatrix} 0 & -0.72 \\ 1 & 1.7 \end{pmatrix}.$$

Then

$$A * A = \begin{pmatrix} 0 & 0.5184 \\ 1 & 2.89 \end{pmatrix}$$

which has spectral radius $\rho = 3.059\ldots$.

However, the set of the (entrywise) nonnegative (resp. symmetric) Schur stable matrices is closed under the Hadamard product. This follows from [12, p. 358, Observation 5.7.4, resp. p. 332, Th. 5.5.1].

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